LEHMANN'S TWO SAMPLE NO-NPARAMETRIC TEST

BY B. V. SUKHATME AND M. V. DESHPANDE
Institute of Agricultural Research Statistics, New Delhi

1. INTRODUCTION AND SUMMARY

The problem of optimum tests was first considered by Neyman and Pearson[1]. Later, Lehmann[2] considered the problem of construction of optimum nonparametric tests against different types of typical nonparametric alternatives. Here we discuss one of the optimum tests proposed by him.

Let $x_1, x_2 \ldots x_m$ and $y_1, y_2 \ldots y_n$ be two independent samples of observations drawn from populations with continuous *cdfs* $F(x)$ and $G(x)$ respectively. For testing that X's are distributed according to F and Y's according to $G=qF+pF^2$, where $o \leqslant p \leqslant 1$ and $p+q=1$, the test criterion for locally most powerful test, derived by Lehmann[2] is given by

$$T=\left\{\sum_{i=1}^{m} r_i-\frac{m(m+n+1)}{2}\right\}^2+\left\{\sum_{i=1}^{n} s_i-\frac{n(n+m+1)}{2}\right\}^2 + \sum_{i=1}^{m}(r_i-i)^2+\sum_{i=1}^{n}(s_i-i)^2$$

where r_i and s_i denote the ranks of x_i and y_i in the sample of $m+n$ observations. The test rejects the null hypothesis when T is too large. It will be seen that the test differs in form, from the two sided Wilcoxons' test in that T has two additional terms. In view of the superiority of T over Wilcoxons' test a detailed study of the proposed test was considered useful. This paper gives an alternative expression for the test statistics and a recurrence relation. Using this relation probability tables for $m \leqslant n \leqslant 6$ are constructed. The first three moments and the limiting distribution of the test statistic under the null hypothesis are worked out. Finally the asymptotic power of the test under normal rectangular and exponential alternatives is calculated for shift in location and compared with the Wilcoxons' two sided test.

2. Proposed Test Statistic and its Exact Small Sample Distribution

Let us denote the ordered x's and y's by $x^1, x^2 \ldots x^m$ and $y^1, y^2, \ldots y^n$ and ranks of x's and y's in the combined sample by $r_1, r_2 \ldots r_m$ and $s_1, s_2, \ldots s_n$ respectively. The proposed test statistic T then rejects null hypothesis when T is greater than or equal to T_o, where $P(T \geqslant T_o) = \alpha$, α being the level of significance.

Now consider the ordered sequence of m x's and n y's. Let μ denote the number of times a y precedes an x, u_1 the number of times a pair of y's precedes on x and u_2 the number of times a pair of x's precedes a y Further let,

$$u' = u_1 + u_2. \qquad \ldots(2.1)$$

Since the rank of ith ordered x *i.e.* x^i is r_i in the combined sample, there will be $r_i - i$ y's preceding x^i. Therefore

$$u = \sum_{i=1}^{m} (r_i - i) = \sum_{i=1}^{m} r_i - \frac{m(m+1)}{2}. \qquad \ldots(2.2)$$

It may be noted that 'u' is same as Wilcoxons' 'u' statistics. It also follows that there will be $\binom{r_i - i}{2}$ number of such pairs of y's preceding x^i. Hence the total number of such pairs will be obtained by summing over all m x's. So that,

$$u_1 = \sum_{i=1}^{m} \binom{r_i - i}{2}$$

$$= \frac{1}{2}\left\{ \sum_{i=1}^{m} (r_i - i)^2 - \sum_{i=1}^{m} (r_i - i) \right\} \qquad \ldots(2.3)$$

Similarly,

$$u_2 = \frac{1}{2}\left\{ \sum_{i=1}^{n} (s_i - i)^2 - \sum_{i=1}^{n} (s_i - i) \right\} \qquad \ldots(2.4)$$

and since,

$$\sum_{i=1}^{m} r_i + \sum_{i=1}^{n} s_i = \frac{(m+n)(m+n+1)}{2} \qquad \ldots(2.5)$$

$$u' = u_1 + u_2$$

$$= \frac{1}{2}\left\{ \sum_{i=1}^{m} (r_i - i)^2 + \sum_{i=1}^{n} (s_i - i)^2 - nm \right\} \qquad \ldots(2.6)$$

it follows

$$T = 2\left[u' + \left(u - \frac{nm}{2} \right)^2 + \frac{nm}{2} \right] \qquad \ldots(2.7)$$

This is an alternative expression for the test statistic which expresses T as a function of generalised 'u' statistics defined by Hoeffding[3].

Let $N_{m,n}(u, u')$ denote the total number of possible sequences of m x's and n y's in each of which y precedes an x u times, and u' is the number of times a pair of x's precedes a y and a pair of y's precedes an x. If the last term in the sequence is an x, u will be decreased by n and u' by $\frac{n(n-1)}{2}$ when the last term is omitted. The number of such sequences, can then be denoted in the above notation by $N_{m-1,n}\left(u-n, u'-n\frac{(n-1)}{2}\right)$. Similarly number of sequences in which the last term omitted is y are, $N_{m,n-1}\left(u, u'-\frac{m(m-1)}{2}\right)$ since the last term in the sequence can be either an x or an y we arrive at the recurrence relation

$$N_{m,n}(u, u') = N_{m-1,n}\left(u-n, u'-\frac{n(n-1)}{2}\right) + N_{m,n-1}\left(u, u'-\frac{m(m-1)}{2}\right) \qquad \ldots(2.8)$$

with,

$$\begin{aligned} N_{m,n}(u, u') &= 0 \text{ if } u \text{ or } u'<0 \text{ ; or } u \text{ and } u'<0 \\ N_{mo}(u, u') &= 0 \text{ if } u \text{ or } u' \neq 0 \\ &= 1 \text{ if } u=u'=0 \\ N_{on}(u, u') &= 0 \text{ if } u \text{ or } u' \neq 0 \\ &= 1 \text{ if } u=u'=0. \end{aligned} \qquad \ldots(2.9)$$

Under the null hypothesis each of $\frac{(m+n)!}{m!\,n!}$ sequences of m x's and n y's is equally likely. Consequently if $P_{m,n}(u, u')$ represent the probability of obtaining u and u' in a sequence of $m+n$ observations, we have

$$P_{m,n}(u, u') = \frac{m}{n+m} P_{m-1,n}\left[(u-n), u'-\frac{n(n-1)}{2}\right] + \frac{n}{n+m} P_{m,n-1}\left[\left(u, u'-\frac{m(m-1)}{2}\right)\right] \qquad \ldots(2.10)$$

The initial conditions of $P_{m,n}(u, u')$ will be similar to those of $N_{m\,n}(u, u')$. Using this recurrence relation probabilities for T have been calculated for $m \leqslant n \leqslant 6$ and presented in table 1. This gives the exact distribution of T for small samples under the hypothesis.

TABLE 1

Probability distribution of T for $m \leqslant n \leqslant 6$

T_o denotes the value of T and P denotes the probability $P\ (T\ T_o)$

T_o	P	T_o	P	T_o	P
n=2	m=2	16	0·733	100	0·200
16	0·333	14	1·000	84	0·257
8	0·667			80	0·343
6	1·000	n=4	m=3	66	0·400
		120	0·029	64	0·486
n=3	m=2	108	0·057	54	0·657
36	0·100	92	0 086	52	0·686
30	0·200	81	0·114	48	0·800
22	0·300	70	0·171	46	0·914
18	0·400	62	0·229	44	1·000
14	0·600	54	0·286		
12	0·900	52	0·314	n=5	m=2
10	1·000	48	0·371	100	0·048
		46	0·400	74	0 095
n=3	m=3	44	0·429	70	0·143
67·5	0·100	40	0·543	54	0·238
47·5	0·200	36	0·629	50	0·286
43·5	0·400	34	0·657	40	0·233
25·5	0·600	32	0·771	38	0·381
23·5	0·700	30	0·971	36	0·476
19·5	1·000	28	1·000	32	0·524
n=4	m=2			30	0·571
32	0·067	n=4	m=4	28	0·714
26	0·133	192	0·029	26	0·810
22	0·267	156	0·057	24	0·857
20	0·400	126	0·114	22	1·000
18	0·467	102	0·171		

(Table 1 contd.)

T_o	P	T_o	P	T_o	P
n=5	m=3	280	0·016	86	0·452
187·5	0·018	254	0 024	84	0·484
157·5	0·036	236	0 032	82	0·500
151·5	0·054	214	0·048	80	0·524
125·5	0·071	198	0·063	78	0·556
121·5	0·107	180	0·079	76	0·587
99·5	0·143	178	0·087	74	0·595
97·5	0·179	166	0·103	72	0·675
95·5	0·196	164	0·111	70	0·698
79·5	0·250	152	0·127	68	0·755
77·5	0·268	148	0·151	66	0·802
75·5	0·321	140	0·167	64	0·889
67·5	0·339	136	0 190	62	0·992
65·5	0·357	130	0·198	60	1 000
61·5	0·429	124	0·214		
69·5	0·482	122	0·238	n=5	m=5
67·5	0·500	120	0·246	437·5	0·008
53·5	0·517	118	0·248	381·5	0·016
51·5	0·536	114	0·262	331·5	0·032
49 5	0·661	112	0·286	287·5	0·048
47·5	0·696	106	0·294	285·5	0·056
45·5	0·732	102	0·341	249·5	0·071
43·5	0·857	100	0·349	245·5	0·096
41·5	0·946	98	0·357	217·5	0·111
39·5	1·000	94	0·405	211·5	0·127
				209·5	0·151
n=5	m=4	92	0·413	183·5	0·167
300	0·008	88	0·429	179·5	0·214

(*Table 1 contd.*)

T_o	P	T_o	P	T_o	P
177·5	0·222	54	0·357	102	0·274
155·5	0·254	52	0·393	100	0·286
153·5	0·278	48	0·464	96	0·333
151·5	0·310	44	0·500	88	0·345
133·5	0·357	42	0·571	86	0·357
131·5	0·373	40	0·607	84	0·405
129·5	0·421	38	0·643	82	0·417
117·5	0·444	36	0·750	80	0·429
113·5	0·516	34	0·786	78	0·488
111·5	0·548	32	0·857	72	0·512
107·5	0·563	30	0·964	70	0·536
101·5	0·587	28	1·000	68	0·571
99·5	0·683	n=6	m=3	66	0·655
97·5	0·690	270	0·012	64	0·667
95·5	0·706	226	0·024	62	0·690
91·5	0·802	216	0·036	60	0·762
89·5	0·889	188	0·060	58	0·833
87·5	0·921	178	0·071	56	0·893
85·5	1·000	156	0·095	54	0·952
n=6	m=2	154	0·107	52	1·000
114	0 036	146	0·131	n=6	m=4
112	0·071	130	0·143	432	0·005
96	0·107	126	0 179	384	0·010
86	0·179	120	0 202	376	0·014
72	0·214	118	0·214	332	0·019
65	0·250	110	0·226	326	0·029
64	0·286	104	0·238	286	0·028

(*Table 1 contd.*)

T_o	P	T_o	P	T_o	P
282	0·048	116	0·424	442	0·024
280	0·052	114	0·433	420	0·028
246	0·062	112	0·462	418	0·030
244	0·076	110	0·467	394	0·035
240	0·090	108	0·524	390	0·041
212	0·105	106	0·543	372	0·045
208	0·119	102	0·562	368	0·052
206	0·129	100	0·571	350	0·056
204	0·143	98	0·586	344	0·061
186	0·148	96	0·629	342	0·067
184	0·152	94	0·700	330	0·071
178	0·167	92	0·710	324	0·076
176	0·181	90	0·733	322	0·082
174	0·210	88	0·757	312	0·084
172	0·214	86	0·838	304	0·089
162	0·219	84	0·914	300	0·112
156	0·224	82	0·952	298	0·104
154	0·229	80	1·000	294	0·106
150	0·267			286	0·110
148	0·286	n=6	m=5	282	0·123
146	0·305	630	0·002	280	0·126
136	0·310	600	0·004	270	0·128
132	0·319	562	0·006	264	0·136
130	0·329	534	0·009	262	0·143
128	0·362	500	0·013	260	0·152
126	0·386	474	0·017	254	0·154
124	0·414	444	0·022	248	0·162

(Table 1 contd.)

T_o	P	T_o	P	T_o	P
246	0·169	156	0·439	784	0·004
244	0·177	154	0·448	710	0·009
234	0·182	152	0·474	642	0·013
230	0·195	150	0·487	640	0·015
228	0·199	148	0·496	580	0·019
226	0·212	146	0·502	576	0·026
220	0·216	144	0 530	524	0·030
216	0 229	142	0·543	518	0·035
214	0·234	138	0·569	516	0·041
212	0·247	136	0·582	474	0·045
204	0·251	134	0·606	466	0·050
202	0·260	132	0·632	462	0·063
198	0·279	130	0·645	420	0·069
196	0·288	128	0 671	414	0 078
192	0·292	126	0·690	412	0·084
190	0·301	124	0·716	410	0·093
186	0·320	122	0·762	372	0·102
184	0·329	120	0·801	368	0·115
180	0·338	118	0·842	366	0·119
176	0·344	116	0·883	364	0·132
174	0·351	114	0·952	330	0·141
172	0·377	112	0·998	328	0·152
170	0·387	110	1·000	324	0·171
166	0·394			322	0·180
164	0·400	n=6	m=6	294	0·193
162	0·429			290	0·206
160	0·431	864	0·002	288	0·212

(*Table 1 concluded*)

T_o	P	T_o	P	T_o	P
286	0·238	212	0·398	168	0·654
284	0·240	206	0·411	166	0·697
264	0·247	204	0·437	164	0·712
258	0·260	202	0·463	162	0·714
256	0·273	200	0·485	160	0·738
254	0·299	190	0·496	158	0·747
252	0·310	186	0·519	156	0·812
240	0·314	184	0·526	154	0·840
232	0·323	182	0·578	152	0·866
228	0·359	180	0·589	150	0·935
226	0·372	174	0·602	148	0·978
224	0·394	172	0·615	146	1·000

3. Moments of T

Since T is a function of u and u' only, all the moments of T can be obtained. The first four moments of u have already been calculated by Mann and Whitney[4]. Proceeding in the same manner and using the recurrence relation given by (2.10) it may be shown that—

$$E(T) = \frac{nm(n+m+1)}{2} \qquad \ldots(3.1)$$

$$= \frac{n^2(2n+1)}{2} \text{ when } m=n \qquad \ldots(3.2)$$

$$V(T) = \mu_2(T)$$

$$= \frac{nm(n+m+1)}{180}\Big[10n^2m + 10m^2n + 14n^2 + 14m^2 - 12nm - 10n - 10m - 16\Big] \qquad \ldots(3.3)$$

$$= \frac{n^2(2n+1)(n^2-1)(5n+4)}{45} \text{ when } m=n \qquad \ldots(3.4)$$

and

$$\mu_3(T)=\frac{nm(n+m+1)}{3780}\Big[140n^2m^2(n^2+m^2)$$
$$+280n^3m^3+147nm(n^3+m^3)-119n^2m^2(n+m)$$
$$-71(n^4+m^4)-37nm(n^2+m^2)-1612n^2m^2$$
$$-16(n^3+m^3)-346nm(n+m)+425(n^2+m^2)$$
$$+442nm+334(n+m)-24\Big] \qquad ...(3.5)$$

$$=\frac{n^2(2n+1)}{945}\Big[140n^6+14n^5-457n^4$$
$$-181n^3+323n^2+167n-6\Big] \text{ where } n=m. \qquad ...(3.6)$$

The calculations become very heavy for higher moments. As such no attempt was made to obtain moments higher than $\mu_3(T)$.

4. Limiting Distribution of T under the Hypothesis $F=G$

Let $\frac{m}{n}\to c>0$ as m and $n\to\infty$

From (2.7) we have

$$\frac{T}{2\mu_2(u)}=\frac{\left(u-\frac{nm}{2}\right)^2}{\mu_2(u)}+\frac{(2u'+nm)}{2\mu_2(u)}. \qquad ...(4.1)$$

It is known from Mann and Whitney that

$$\frac{u-nm/^2}{\sqrt{\mu_2(u)}}$$

is asymptotically normally distributed with mean 0 and variance 1. It follows that the first term is (4.1) in asymptotically distributed as x^2 with 1 *d.f.*

Now, from the results obtained earlier it can be seen that,

$$E\,\frac{(2u+nm)}{2\mu_2(u)}=2. \qquad ...(4.2)$$

and

$$V\left(\frac{(2u'+nm)}{2\mu_2(u)}\right)=\frac{36}{45}\left[\frac{4(n-m)^2+(n+m-2)\delta}{nm(n+m+1)}\right]$$

which tends to zero as m and n tend to infinity. It follows that the second term in (4.1) tends in probability to 2. $\frac{T}{2V(u)}-2$ is therefore asymptotically distributed as x^2 with 1 *d. f.* This can also be seen from the fact that

$$\beta_1 (T)=\frac{\mu_2^3 (T)}{\mu_2^3 (T)}$$

tends to 8 as m and n tend to infinity. The values of $\beta_1 (T)$ given below will indicate how the distribution of T approaches to x^2 distribution with increasing values of n $(m=n)$.

$n=$	8	10	15	20	30	40	60	100
$\beta_1(T)=$	5·5	5·9	6·6	6·9	7·2	7·5	7·6	7·8

5. Asymptotic Power of the Test

It has already been shown that T can be expressed as a function of u and u' and is given by—

$$T=2\left(u-\frac{nm}{2}\right)^2+2u'+nm.$$

The means, variances and covariance of u and s under the alternative hypothesis are given by

$$E (u)=n^2m_{10} \qquad \text{...(5.1)}$$

$$E (s)=n^3+n^2 (n-1) (m_{20}-2m_{11}). \qquad \text{...(5.2)}$$

$$V (u)=n^2m_{10}+nn^{(2)} (m_{20}+2m_{10}-2m_{11}) +n^{(2)}n^{(2)}m^2{}_{10}-(n^2m_{10})^2 \qquad \text{...(5.3)}$$

$$V (s) =nn^{(2)} [-2 (2n^2-4n+1) m_{20}+4 (n^2-n-1)m_{11} +4 (n8-2) (m_{30}-3m_{12})+(n-2) (n-3) (m_{40}-4m_{13}) +(n-1) (n-2) (-4m_{31}+8m_{22}+4A_1+8A_2) -(n-1) (5n-6) (m^2{}_{20}+4m_{11}{}^2)+16 (n-1)^2(m_{11}m_{20})] \qquad \text{...(5.4)}$$

$$\text{Cov} (u, s)=nn^{(2)} [n (2m_{20}-m_{10})+(n-2) (3m_{12} -2m_{11}+m_{30})-4 (n-1)m_{21} +(3n-2)m_{10} (2m_{11}-m_{20})] \qquad \text{...(5.5)}$$

where, $S=2 u'+nm$

$$m_{ij}= \int F^i(x) G^j(x) dG(x)$$

$$n^{(i)}=n (n-1) \dots (n-i+1)$$

$$A_1= \iint\limits_{y_i\leqslant y_k} F^2 (y_i) F(y_k) dG(y_i) dG(y_k)$$

$$A_4= \iint\limits_{y_i\leqslant y_k} F(y_i) G(y_i) F(y_k) dG(y_i) dG(y_k).$$

Using the fact that the joint asymptotic distribution of u and u' with proper devisors is bivariate normal the power of the test has been calculated by numerical integration for different values of θ where $F(x)=G(x+\theta)$ in case of samples drawn from normal, exponential and rectangular populations with $m=n=60$. Corresponding power has also been obtained for the Wilcoxons test. These results are given in table 2.

TABLE 2

Comparison of the Wilcoxon test and the T test.

Level of significance=0·05

Type of Alternative	θ	*Power of Wilcoxons' Test*	*Power of T test*
	0·10	0·082	0·085
	0·20	0·183	0·195
	0·30	0·352	0·366
	0·40	0·557	0·564
Normal	0·50	0·753	0·756
	0·60	0·890	0 891
	0·80	0 991	0·991
	0·03	0·085	0·096
	0·05	0·149	0·171
	0·07	0·244	0·257
Rectangular	0·10	0·432	0·437
	0·15	0·758	0·767
	0 20	0·944	0·945
	0·10	0·075	0·088
	0·20	0·155	0·187
	0·30	0·285	0·299
	0·40	0·454	0·463
Exponential	0·50	0·631	0·635
	0·60	0·782	0·785
	0 80	0·952	0·953
	1·00	0·995	0·995

The power of the proposed test under normal, rectangular and exponential alternatives for shift in location is seen to be greater than that of Wilcoxon's two sided test. The difference between powers is more when θ is small and decreases as θ increases. For large values of θ the difference is almost negligible as is to be expected.

ACKNOWLEDGEMENTS

We are thankful to Dr. V. G. Panse, Statistical Adviser, Indian Council of Agricultural Research for providing the facilities during the preparation of this paper.

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